Lecture 12 (Basics of graph theory)

1 Basic Terminologies

Definition 1 A graph G consists of two sets V and E, where V is a non-empty set, called the vertex set, and E is called the edge set. A graph G is also denoted as $G = (V, E)$. We define some terminologies in a graph G as follows.

1. The number $|V|$ is called the order of the graph G and is denoted as $|G|$. By $||G||$, we denote the number of edges in G.

2. A graph is non-trivial if it has at least one edge; otherwise, it is called a trivial graph.

3. An edge $e \in E$ joining vertices u and v is denoted as $e = uv$. In this case, u and v are called adjacent vertices, also called the end vertices, of the edge e. We say that an edge e is incident on a vertex u if u is an end vertex of the edge e. If two vertices u and v are adjacent, we denote $u \sim v$; otherwise, $u \not\sim v$. Two edges $e_1, e_2 \in E$ are called adjacent if they have a common end vertex.

4. If the end vertices of an edge $e \in E$ are the same, then the edge is called a loop. If e_1, e_2 are two edges such that they have the same end vertices, then the edges are called parallel edges or multiple edges. A graph is called simple if it has no loops or multiple edges. Unless stated otherwise, by a graph, we mean a simple graph.

5. For a vertex $v \in V$, the neighborhood $N(v)$ of v is $N(v) = \{u \in V : u \sim v\}$. If $A \subseteq V$, then $N(A)$ is defined as $N(A) = \cup_{v \in A} N(v)$.

6. The degree of a vertex v in a graph G is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

7. the minimum degree of a vertex in G is denoted by $\delta(G)$, and the maximum degree of a vertex in G is denoted by $\Delta(G)$.

8. A set of vertices or edges is said to be independent if no two of them are adjacent. The

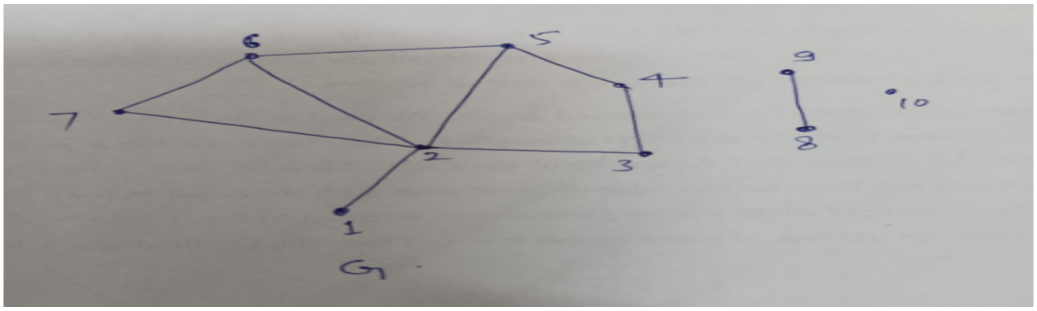

Figure 1:

maximum size of an independent vertex set is called the independence number of G, denoted $\alpha(G)$.

Example: Consider the graph G, given in Figure 1. The vertices 1, 8, and 9 are pendent vertices. The vertex 10 is isolated. Since $N(2) = \{1, 3, 5, 6, 7\}$, $\deg(2) = 5$. Clearly $\delta(G) = 0$ and $\Delta(G) = 5$. Note that $\alpha(G) = 6$, as the set $\{1, 3, 5, 7, 9, 10\}$ is a maximum independent set.

Definition 2 Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Then

1. G is called *complete graph*, denoted as K_n, if every pair $\{v_i, v_j\}$, $i \neq j$, forms an edge.
2. G is called *path graph*, denoted as P_n, if $E(G) = \{v_i v_{i+1} \,:\, 1 \leq i \leq n-1\}$.
3. G is called *cycle graph*, denoted as C_n, $n \geq 3$, if $E(G) = \{\{v_i v_{i+1} \,:\, 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.
4. G is called *bipartite graph* if $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$ and each edge $e \in E$ has one end vertex in V_1 and other in V_2.
5. G is called *complete bipartite graph* if G is a bipartite graph with partite sets V_1 and V_2 such $u \sim v$, $\forall\, u \in V_1$ and $u \in V_2$. If $|V_1| = m$ and $|V_2| = n$, G is denoted by $K_{m,n}$.

Question. What is the maximum number of edges possible in a simple graph of order n?

Lemma 1 [**Handshaking Lemma:**] Let $G = (V, E)$ be any graph (need not be simple), then

$$\sum_{v \in V} \deg(v) = 2|E|$$

.

Proof: Note that each edge contribute 2 to the sum $\sum_{v \in V} \deg(v)$. Hence, $\sum_{v \in V} \deg(v) = 2|E|$.

The degree of a vertex is considered as 2 if it has a loop.

Corollary 1 *Let $G = (V, E)$. Then the number of odd-degree vertices is even.*

Proof: Follows from the above lemma.

Question. At a party, each person shakes hands with some others. If the total number of handshakes is 10.

1. What is the sum of the degrees of all people?
2. Can there be exactly 3 people with odd number of handshakes? Justify.

Solution. Think of people as vertices, handshakes as edges, and the number of handshakes a person has is their degree.
(1) By the Handshaking Theorem: sum of degrees $= 2\times$ (number of handshakes) $= 20$.
(2) From the Handshaking Theorem: the number of vertices (people) with odd degree is always even. So: 3 is an odd number Hence, it is not possible.

Question. In a party, total handshakes $= 15$. One person shook hands with everyone else. How many people are there?
Solution. Let total people $= n$. Then the degree of each is $n-1$. So degree sum $= n \times (n-1)$. Thus the number of edges (handshakes) $= n(n-1)/2 = 15$. This implies $n = 6$.

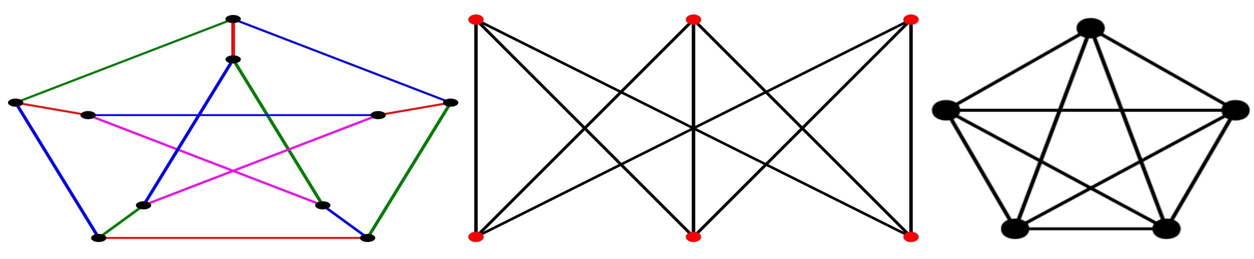

Figure 2: Petersen graph, Complete bipartite graph $K_{3,3}$, Complete graph K_5

Proposition 1 In a graph $G = (V, E)$ with $|V| = n \geq 2$, there are two vertices of equal degree.

Proof: If G has two or more isolated vertices, then we are done. Suppose G has one isolated vertex. Then the remaining $n-1$ vertices have degree between 1 to $(n-2)$ and hence by the Pigeonhole Principle (PHP), the result holds. Otherwise, G has no isolated vertices. Then there are n vertices whose degrees lie between 1 to $n-1$. Again by PHP, the result holds.

Theorem 1 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: First assume that $G = (V, E)$ is bipartite graph with bipartite subsets V_1 and V_2 of V. Then assign one color to each vertex of V_1 and a second color to each vertex of V_2 will give the desired condition.

Conversely, let it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. $\square$

Definition. Let $G = (V, E)$ be a graph. Then G is called k-regular if $\deg(v) = k$ for all $v \in V$. A 3-regular graph is called cubic.

Necessary and sufficient condition A k-regular simple graph on n vertices exists iff kn is even and $n \geq k + 1$.

Solution. ($\Rightarrow$) By Handshaking lemma, the degree sum $= k \times n = 2|E|$, which is even. Also, $k \leq n - 1$, which means $n \geq k + 1$.
($\Leftarrow$) Assume nk is even and $n \geq k + 1$. We show that there is a k-regular graph on n vertices.

Case 1: Suppose k is even. Arrange the n vertices in a circle, and then connect each vertex to $k/2$ nearest neighbors on each side.

Case 2: Suppose k is odd. Since kn is even and k is odd, n must be even. Now arrange each vertex in a circle and connect each vertex to $(k - 1)/2$ nearest vertices on each side. Then connect each vertex to the vertex directly opposite in it (the vertex $n/2$ steps away).

Example.

1. The graph P_4 is not regular.
2. The cycle graph C_n is 2-regular whereas the complete graph K_n is $(n - 1)$-regular.
3. The Petersen graph, K_4, and $K_{3,3}$ are cubic.

2 Construction of new graphs from existing graphs

Definition 3 Let $G = (V(G), E(G))$ be a graph. Then

1. A graph $H = (V(H), E(H))$ is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
2. A subgraph $H = (V(H), E(H))$ of G is called a spanning subgraph of G if $V(H) = V(G)$.
3. A subgraph $H = (V(H), E(H))$ of $G = (V, E)$ is called an induced subgraph of G if for every $u, v \in V(H)$, $e = uv \in E(H)$ whenever $e = uv \in E(G)$.
4. If $v \in V$, then the graph $G - v$, called the *vertex deleted subgraph*, is obtained from G by deleting v and all the edges that are incident with v.
5. If $e \in E$, then the graph $G - e = (V, E \setminus \{e\})$ is called the *edge deleted subgraph*.
6. If $u, v \in V$, then $G + uv = (V, E \cup \{uv\})$ is called the graph obtained by edge addition.
7. The complement $\overline{G}$ of G is defined as $(\overline{V}, \overline{E})$, where $\overline{V} = V$ and $\overline{E} = \{uv \mid u \neq v, uv \notin E\}$.

Question. Consider a graph G with n vertices. Then

1. $||G|| + ||\bar{G}|| = ^{|G|} C_2$.
2. $\deg_G(v) + \deg_{\bar{G}}(v) = n - 1$. Thus $\Delta(G) + \Delta(\bar{G}) \geq n - 1$.

Definition 4 Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

1. Then their intersection, denoted $G \cap H$, is defined as $(V(G) \cap V(H), E(G) \cap E(H))$.
2. Then their union, denoted $G \cup H$, is defined as $(V(G) \cup V(H), E(G) \cup E(H))$.
3. Then their Cartesian product, denoted $G \times H$, has $V(G) \times V(H)$ as the vertex set and the edge set consists of all elements $\{uv, u'v'\}$, whether either $u = u'$ and $\{v, v'\} \in E(H)$, or $v = v'$ and $\{u, u'\} \in E(G)$.

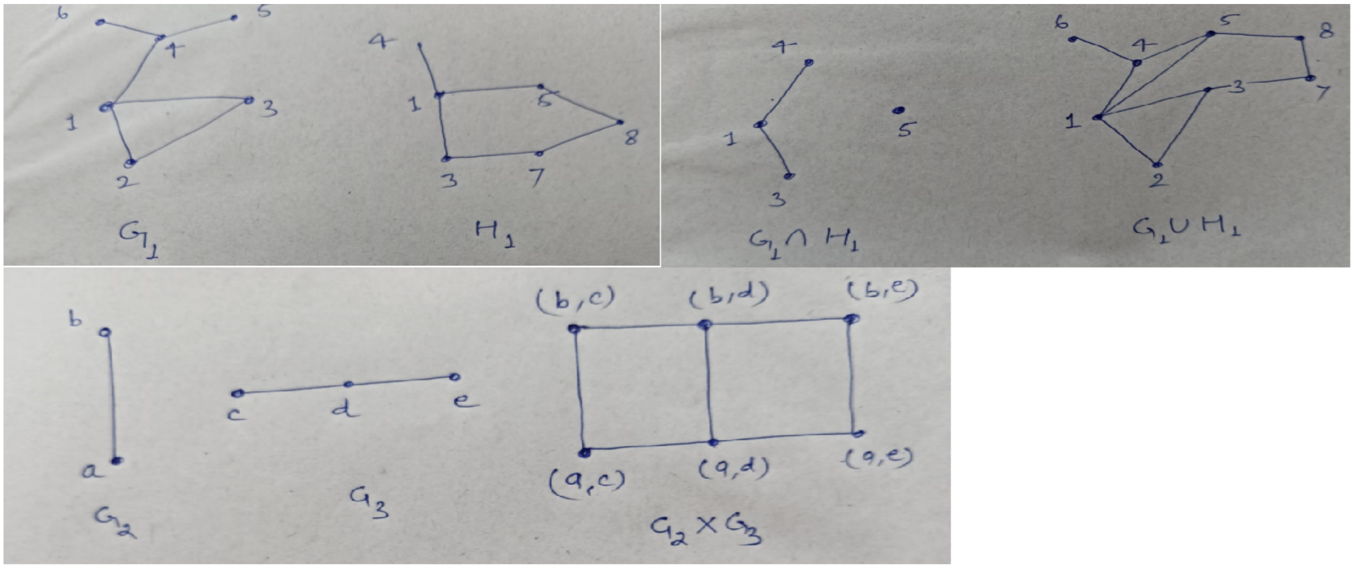

Figure 3: Intersection, Union, and Product of graphs

References:

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2: K. H. Rosen; Discrete mathematics and its applications, Tata McGraw-Hill.